Exact solutions to non-linear chiral field equations

1976 J. Phys. A: Math. Gen. 9 1331

(http://iopscience.iop.org/0305-4470/9/8/023)

J. Phys. A: Math. Gen., Vol. 9, No. 8, 1976. Printed in Great Britain.

Exact solutions to non-linear chiral field equations

John M Charap

Department of Physics, Queen Mary College, Mile End Road, London E1 4NS, UK

Received 8 March 1976

Abstract. Exact solutions of the classical non-linear field equations for the chiral-invariant model of pion dynamics are presented. The solutions depend on the space–time variable x^μ through $k_\mu x^\mu$, where k_μ is an arbitrary constant 4-vector.

This brief paper is prompted by the present interest in obtaining solutions to classical non-linear field equations. One such equation arises in the study of chiral-invariant pion models. If $\phi^i(x^\mu)$ $(i = 1, 2, 3)$ denotes the isotriplet of pion fields, the Lagrange density may be written

$$L = \tfrac{1}{2}\partial_\mu\phi^i g_{ij}(\phi)\partial^\mu\phi^j, \tag{1}$$

where $\mathrm{d}\phi^i g_{ij}\,\mathrm{d}\phi^j$ is as usual the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ invariant metric. The form of g_{ij} depends on the particular choice made for the chiral transformations of ϕ^i. In general, the field equations which result from the Lagrangian (1) are

$$\Box\phi^i + \Gamma^i{}_{jk}\partial_\mu\phi^j\partial^\mu\phi^k = 0. \tag{2}$$

Here the Christoffel symbol $\Gamma^i{}_{jk}$ is given by

$$g_{li}\Gamma^i{}_{jk} = \tfrac{1}{2}(g_{lj,k} + g_{lk,j} - g_{jk,l}), \tag{3}$$

and $g_{lj,k} \equiv \partial g_{lj}(\phi)/\partial\phi^k$, etc. For the particular choice of chiral transformation

$$\delta_5^i\phi^j = f_\pi(\delta^{ij} + \phi^i\phi^j/\phi^2), \tag{4}$$

which we have elsewhere (Charap 1973) called the tangential parametrization, the Christoffel symbol takes the simple form

$$\Gamma^l{}_{ij} = -(f_\pi^2 + \phi^2)^{-1}(\delta_{ik}\delta^l_j + \delta_{jk}\delta^l_i)\phi^k, \tag{5}$$

so that the field equations read

$$\Box\phi^i = \partial_\mu\phi^i\partial^\mu[\ln(f_\pi^2 + \phi^2)]. \tag{6}$$

(We have written ϕ^2 for $\Sigma_{i=1}^3(\phi^i)^2$ in the above equations: f_π is a constant.) It is these non-linear field equations which we consider, and for which we obtain a class of solutions.

Our method is to look for solutions to (6) for which the fields ϕ^i depend on the field-point x^μ only through $k_\mu x^\mu$, where k_μ is an arbitrary constant Lorentz 4-vector. Then, if a prime is used to denote differentiation with respect to $k_\mu x^\mu$, the partial differential equations (6) become the ordinary differential equations

$$k^2(\phi^i)'' = k^2(\phi^i)'[\ln(f_\pi^2 + \phi^2)]'. \tag{7}$$

This is just the set of equations which arise in the mechanical analogue to the field theory, when one has an isotriplet of dynamical variables $q^i(t)$, which depends only on time, not on space–time. Their solution is given (Charap 1973) in terms of constant isovectors $\boldsymbol{A}$ and $\boldsymbol{V}$ which satisfy

$$\boldsymbol{A}\,.\,\boldsymbol{V}=0, \qquad \boldsymbol{V}^2+\boldsymbol{A}^2=f_\pi^4. \tag{8}$$

If $\boldsymbol{\phi}$ is the isovector of fields with components ϕ^i which satisfy (7), and hence (6), we have

$$\boldsymbol{\phi}=f_\pi[\boldsymbol{V}\times\boldsymbol{A}-f_\pi^2\boldsymbol{A}\tan(k_\mu x^\mu)]/A^2. \tag{9}$$

This is a solution for arbitrary k_μ.

The axial and vector currents associated with the solution (9) are easily found to be respectively

$$\boldsymbol{A}_\mu=\boldsymbol{A}k_\mu, \qquad \boldsymbol{V}_\mu=\boldsymbol{V}k_\mu. \tag{10}$$

Since they are constant, they are trivially conserved; since they are parallel in Lorentz space, their covariant curls also trivially vanish. A less fortunate consequence of their being constant is that the associated charges are either infinite or zero.

The energy–momentum tensor $T_{\mu\nu}$ is also easily evaluated; it is

$$T_{\mu\nu}=f_\pi^2(k_\mu k_\nu-\tfrac{1}{2}k^2\eta_{\mu\nu}), \tag{11}$$

again constant.

Having obtained the solution in one parametrization, it is easy to go to any other. For example, in the σ-model parametrization, for which

$$\delta_5^i\phi^j=(f_\pi^2+\phi^2)^{1/2}\delta^{ij}, \tag{12}$$

the field equations are

$$\Box\boldsymbol{\phi}+f_\pi^{-2}\boldsymbol{\phi}[\partial_\mu\boldsymbol{\phi}\,.\,\partial^\mu\boldsymbol{\phi}+(\boldsymbol{\phi}\,.\,\partial_\mu\boldsymbol{\phi})(\boldsymbol{\phi}\,.\,\partial^\mu\boldsymbol{\phi})(f_\pi^2-\phi^2)^{-1}]=0, \tag{13}$$

to which the solution is

$$\boldsymbol{\phi}=(\boldsymbol{V}\times\boldsymbol{A}\cos kx-f_\pi^2\boldsymbol{A}\sin kx)/(Af_\pi). \tag{14}$$

Of course equations (10) and (11) still hold, as they are independent of parametrization.

Reference

Charap J M 1973 *J. Phys. A: Math., Nucl. Gen.* **6** 987